

Generalized Hot Enhançons

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ABSTRACT: We review what has been learnt and what remains unknown about the physics of hot enhançons following studies in supergravity. We recall a rather general family of static, spherically symmetric, non-extremal enhançon solutions describing D4 branes wrapped on K3 and discuss physical aspects of the solutions. We embed these solutions in the six dimensional supergravity describing type IIA strings on K3 and generalize them to have arbitrary charge vector. This allows us to demonstrate the equivalence with a known family of hot fractional D0 brane solutions, to widen the class of solutions of this second type and to carry much of the discussion across from the D4 brane analysis. In particular we argue for the existence of a horizon branch for these branes.

KEYWORDS: Black Holes in String Theory, D-branes.

JHEP10(2006)024

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1. Introduction

The celebrated AdS/CFT correspondence [1] has led to significant advances in the study of string theory and strongly coupled gauge theories. The original analysis concerned maximally supersymmetric super-Yang Mills (SYM) theories but was soon extended to more general gauge theories.

Our focus here is on brane systems giving rise to SYM theories at large N with $\mathcal{N} = 2$ supersymmetry and no hypermultiplets. For these systems, there are strong hints that aspects of the strong-coupling behaviour of the SYM theories can be understood from supergravity, despite the lack of a strong/weak duality in the decoupling limit. This was first shown for the enhançon system [2].

Going to finite temperature can yield important new information about the nature of dualities obtained via the decoupling limit from systems of branes. For the $\mathcal{N} = 4$ $SU(N)$ SYM theory in four dimensions, this was demonstrated in [3] where various aspects of the finite temperature gauge theory at large N were found to be reproduced by the supergravity dual.

Studies of finite temperature enhançon systems exist in the literature [4–6] and we proceed to review the physics of these systems in section 1, adding some new remarks. We review the evidence for a novel kind of finite temperature phase transition in this class of theories.

In section 2, we present a general family of supergravity solutions for wrapped branes in type IIA on K3 with charges constrained such that enhançon behaviour can occur. We

also tidy up the literature, by demonstrating the equivalence between solutions representing D4 branes wrapped on K3 and fractional D0 branes on the T^4/\mathcal{Z}_2 orbifold limit of K3. We motivate why black hole uniqueness theorems likely specify the physics of the general horizon branch completely. We also broaden the class of shell branch solutions on the fractional brane side.

In section 3, we perform some explicit translations of hot enhançon physics into fractional brane language to resolve a puzzle in the literature. Section 4 contains some open problems and speculations about future directions.

2. Observations on previous related work

The enhançon system was the first setup in which a supergravity dual of pure $\mathcal{N} = 2$ SYM theory with no hypermultiplets was studied [2]. It was constructed by wrapping BPS D-branes on a K3 manifold, and studying the resulting geometry. From the supergravity point of view, the system exhibited a novel singularity resolution mechanism. Naively, there appeared to be a naked timelike singularity in the space transverse to the branes, dubbed the repulson, because a massive particle would feel a repulsive potential which becomes infinite in magnitude at a finite radius from the naive position of the branes. Probing the background with a wrapped D-brane, however, showed that the N source D-branes do not, in fact, sit at the origin. Rather, they expand to form a shell of branes, inside of which the geometry does not, after all, become singular.

In the original enhançon case, taking the decoupling limit did not result in a clean duality, in the sense that the supergravity dual of the strongly coupled gauge theory is not weakly coupled. Nonetheless, a strong hint of the gauge dual of the enhançon mechanism was seen, in terms of nonperturbative corrections to the moduli space of pure $\mathcal{N} = 2$ gauge theory. (Corrections due to finite- N were not ascertained in the supergravity picture, which was studied without loop corrections.)

2.1 Heating up the enhançon system

A natural generalisation was to study enhançon geometries for which the system gains energy above the BPS bound. An unusual two-branch structure was found [2, 4]. One class of possible solutions had the appearance of a black hole (or black brane), and was dubbed the horizon branch, while the other appeared to have an enhançon-like shell surrounding an inner event horizon and was dubbed the shell branch. Only the shell branch correctly matches onto the BPS enhançon solution in the limit of zero energy above extremality but, for sufficiently high extra energy, both solutions were seen to be consistent with the asymptotic charges. The presence of the horizon branch far from extremality was expected, since there, the system should look like an uncharged black hole, when the energy is highly dominant over the charge. Additionally, for the shell branch, fixing the asymptotic charges did not specify exactly how the extra energy distributed itself between the inner horizon and the shell.

Dimitriadis and Ross did a preliminary search [7] for a classical instability that would provide evidence that the two branches are connected. Such an instability, which is funda-

mentally different in nature from the Gregory-Laflamme instability, could be interpreted as signalling a phase transition in the dual gauge theory. Such instability was not found. Also presented was an entropic argument that, at high mass, the horizon branch should dominate over the shell branch in a canonical ensemble. In later work [8], a numerical study of perturbations of the non-BPS shell branch was completed, but still no instability was found. An analytic proof of non-existence of such instabilities could not be found either, owing to the non-linearity of the coupled equations. Furthermore, [8] investigated whether the shell branch might violate a standard gravitational energy condition. Indeed, they found that the shell branch violates the weak energy condition (WEC). This matter will be important for us in a later section, and so we review it here.

In general, the WEC demands that $T_{\mu\nu}v^\mu v^\nu \geq 0$, where v^μ is any timelike vector. For static geometries such as the heated-up enhançon, this condition reduces to

$$\rho \geq 0, \quad \rho + P \geq 0, \tag{2.1}$$

where ρ is the energy density and P is the pressure. The shell branch solution for the hot enhançon system has N source branes located, owing to the enhançon mechanism, at an incision radius r_i rather than at $r = 0$. Because of the shell, the supergravity fields are not differentiable at the incision radius; the Israel jump conditions produce the required stress tensor of the shell of branes (and their excitations). Picking the system of D4-branes wrapped on K3, for definiteness, the energy density of this system at r_i has the form

$$\rho \sim -\frac{Z'_0}{Z_0} - \frac{Z'_4}{Z_4} + \frac{8}{r_i} \left(\sqrt{\frac{L}{K}} - 1 \right). \tag{2.2}$$

In this expression the harmonic functions Z_0, Z_4 are the usual ones exterior to D4-branes outside the shell; Z_0, Z_4 are just constant in the interior. Also, the functions K and L parameterize the non-extremality exterior and interior to the shell, respectively. L is a constant if the interior is flat space; by Gauss' law, the other options are to have a dilaton black hole inside and/or a hot gas. Now, to avoid unnecessarily complicating the analysis, we will take the interior to be flat space. It would be possible to paste in a dilaton black hole instead. The jump conditions tell us that we will put the least stringent constraints on the shell branch supergravity solutions by taking flat space inside. We do this in what follows.

Surprisingly, when the system is near extremality and the asymptotic volume of the K3 is large, the first two terms combine into a dominant, negative, contribution. Thus the shell branch violates the WEC. It was argued [8] that the shell branch should therefore be regarded as unphysical. Accordingly, the horizon branch should be considered the dominant, valid, supergravity solution for non-BPS enhançons, for the range of parameters admitting it. For the region of parameter space in which no horizon branch exists, other solutions, more general than those yet considered, might be valid [8].

In subsequent work on non-BPS enhançons, involving two of the current authors, we used simple supergravity techniques to find the most general solutions with the correct symmetries and asymptotic charges of the hot enhançon system [5]. We showed that the only non-BPS solution with a well-behaved event horizon is the horizon branch.

We also found that there exists a class of solutions that are generalizations of the shell branch. An example of such a generalization is a two-parameter family, dubbed ‘ κ -shell solutions’, of which the old shell branch is a one-parameter subset. Part of this family actually obeys the weak energy condition, and is therefore a candidate for the correct physical solution. Demanding that the WEC be satisfied, however, only fixes κ to obey an inequality. Since we no longer had a microscopic description of the non-BPS geometry, and therefore could not rely upon the supergravity solution being built solely out of D-branes, we could not use a D-probe analysis to distinguish which of these solutions is the correct generalization of the shell branch.

A further few comments on our general solutions are in order here. The general $D = 10$ solution for non-BPS D4-branes wrapped on a K3 (of volume V at infinity) are:

$$\begin{aligned}
 dS_{10}^2 &= -\frac{e^{2a-6c}}{e^{\frac{1}{2}(X_0+X_4)}} dt^2 + e^{2c+\frac{1}{2}(X_0+X_4)} (dR^2 + R^2 d\Omega_4^2) + e^{\frac{1}{2}(X_0-X_4)} ds_{K3}^2, \\
 4\Phi &= 3X_0 - X_4, \\
 F_{(4)} &= Q_4 \epsilon_{S^4}, \\
 F_{(8)} &= q_4 \epsilon_{S^4} \wedge \epsilon_{K3},
 \end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
 e^a &= \left(1 - \frac{r_H^6}{R^6}\right), \\
 e^{3c} &= \left(1 + \frac{r_H^3}{R^3}\right)^2 \left(\frac{R^3 + r_H^3}{R^3 - r_H^3}\right)^{A_1}, \\
 e^{X_0} &= \left(\frac{R^3 - r_H^3}{R^3 + r_H^3}\right)^{-\kappa} \left(\beta - \frac{q_4^2}{144r_H^6(A_1 + \kappa + 1)^2} \left(\frac{R^3 - r_H^3}{R^3 + r_H^3}\right)^{2(A_1 + \kappa + 1)}\right), \\
 e^{X_4} &= \left(\frac{R^3 - r_H^3}{R^3 + r_H^3}\right)^{-\gamma} \left(\alpha - \frac{Q_4^2}{144r_H^6(A_1 + \gamma + 1)^2} \left(\frac{R^3 - r_H^3}{R^3 + r_H^3}\right)^{2(A_1 + \gamma + 1)}\right),
 \end{aligned} \tag{2.4}$$

where $r_H^3 \geq 0$ and asymptotic flatness implies that

$$\begin{aligned}
 \alpha &= \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{Q_4^2}{36r_H^6(A_1 + \gamma + 1)^2}}, \\
 \beta &= \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{q_4^2}{36r_H^6(A_1 + \kappa + 1)^2}}.
 \end{aligned} \tag{2.5}$$

Q_4 is the D4-brane charge and q_4 is the induced D0-brane charge and is related to the D4-brane charge by $q_4 = -V_* Q_4 / V$. Notice that there are four parameters in these solutions: $r_H^3, \kappa, \gamma, A_1$. We must determine which ranges of parameters give physically interesting geometries.

The first condition we demand is that these geometries actually possess an enhançon. To find enhançons, we can study a wrapped D4-brane probe, which takes the form (in static gauge)

$$S_{\text{probe}} = - \int dt m(R) \sqrt{-\mathbb{P}(g)} e^{-\Phi} + \mu_4 \int \mathbb{P}(C_{(5)}) - \mu_0 \int \mathbb{P}(C_{(1)}). \tag{2.6}$$

where the (local) mass of the probe is

$$m(R) = \mu_4 V(R) - \mu_0. \tag{2.7}$$

$V(R) = V e^{X_0 - X_4}$ is the volume of the K3 at a radius R , and the ratio of the D0- and D4-brane charges of the probe is $\mu_0/\mu_4 = V_*$. The probe action breaks up, as usual, into potential and kinetic pieces. The potential terms fail to cancel, owing to breaking of supersymmetry. An enhançon occurs when the probe becomes massless, i.e. satisfies

$$e^{X_0 - X_4}|_{R_e} = \frac{V_*}{V}. \tag{2.8}$$

As an aside, we can also probe with an ordinary D0-brane and get the expected result: the D0-brane can pass right through the enhançon radius.

In order to simplify the relevant expressions for understanding the enhançon condition, let us define the following shorthand,

$$\zeta \equiv (A_1 + \gamma + 1), \quad \eta \equiv (A_1 + \kappa + 1). \tag{2.9}$$

The volume of the K3 varies with radius, as we come in from infinity. We find that there are five different cases depending on the values of ζ and η . In particular:

Case I: $\eta > 0$ and $\zeta \geq 0$

Here, the story is particularly simple. We find that, at some radius greater than r_H , the volume of the K3 always shrinks to zero, indicating that somewhere outside this radius, the K3 has reached its stringy volume. Note that the old ($A_1 = 0 = \kappa = \gamma$) shell solution [4] falls into this category.

Case II: $\eta > 0$ and $\zeta < 0$

This is more complicated. Here, the K3 volume is a ratio of functions which both have zeroes at some finite distance outside r_H . If the denominator wins this competition, the K3 decompactifies at a finite radius rather than developing stringy volume appropriate to the enhançon. Otherwise, i.e. if the numerator wins, there will be an enhançon shell: the K3 shrinks down to its stringy volume.

The condition to get an enhançon rather than a decompactification is

$$\left(\frac{\beta}{\beta - 1}\right)^{1/2\eta} > \left(\frac{\alpha - 1}{\alpha}\right)^{1/2|\zeta|}. \tag{2.10}$$

Case III: $\eta \leq 0$ and $\zeta < 0$

The K3 volume always blows up at a finite radius. None of the Case III solutions has an enhançon and they are all expected to be unphysical.

Case IV: $\eta \leq 0$ and $\zeta \geq 0$

In this case, the physics depends on the ratio $|\zeta/\eta|$. When this ratio is (strictly) less than unity, the volume of the K3 shrinks to zero at r_H , passing through the stringy volume just outside this, where the enhançon lives. Conversely, when this ratio is (strictly) greater than unity, the K3 decompactifies at r_H and so this is not really a shell-branch solution. There is a third, special, case when η and ζ are both zero. For this geometry, there are

significant simplifications, and we find the surprising fact that the K3 volume does not run at all coming in from infinity. Clearly, then, this does not have an enhançon either. Now, since $\zeta = 0$ and $\eta = 0$, there is only one remaining parameter which we can choose to be κ . In fact, we can show that these solutions are unphysical regardless of the value of κ , but the reason differs depending on κ . Either the metric is singular and the dilaton blows up, or the solution violates the BPS bound.

All physical supergravity solutions must obey the BPS bound. In our case, this inequality is

$$M \geq M_{BPS} = \frac{\Omega_4}{16\pi G_6} (|Q_4| - |q_4|), \quad (2.11)$$

where

$$M = \frac{3\Omega_4 r_H^3}{4\pi G_6} \left((A_1 + 1)(\alpha + \beta - \frac{2}{3}) + \gamma\alpha + \kappa\beta - \frac{1}{2}(\kappa + \gamma) \right). \quad (2.12)$$

This puts a further constraint on the physically admissible values of the parameters (r_H^3 , A_1 , γ , κ).

Another condition that physical enhançon solutions should obey is that the WEC be satisfied at the location of the enhançon shell. This will give us another (different) inequality that the parameters must satisfy. Note that knowledge of the microscopic description of our shell could be expected to tie down all four parameters, either partially or completely. Now, the general WEC at the shell is a messy expression; to clarify the physics, let us study a simpler subclass of this solution space.

To illustrate, let us consider the subclass where we set $A_1 = 0 = \gamma$. We will call these the κ -shell solutions. They are a two-parameter family of solutions obeying two inequalities (the BPS bound and the WEC at the shell). For fixed charges and mass above extremality, we can take κ to be the independent parameter. The two inequalities restrict the range of κ . This range depends on the mass above extremality; in the BPS limit, the allowed range expands to include $\kappa = 0$, which corresponds to the known BPS enhançon solution. In the non-BPS case, some of the range of κ satisfying the WEC at the shell and the BPS bound might not be physical either; however, we do not have a microphysical model to settle this question definitively.

It is straightforward to find an expression for the enhançon radius of the κ -shell solutions:¹

$$R_e^3 = 2(\kappa + 1)r_H^3 + \frac{2}{(V - V_\star)} \left(V_\star \sqrt{\frac{Q_4^2}{36} + r_H^6} + V \sqrt{\frac{q_4^2}{36} + r_H^6(\kappa + 1)^2} \right), \quad (2.13)$$

It is clear that the size of any enhançon shell must be larger, for a given fixed mass, than the size of the black hole on the horizon branch, because otherwise the second law of thermodynamics would be violated.

¹We could also rewrite this in terms of the parameters: $Q_4 = -3R_4^3$, $q_4 = -3R_0^3$, $r_H^3 = \frac{1}{4}r_0^3$, in order to put the solution exactly in terms of the language of previous studies [4].

Later, it will be useful to have these solutions in a Schwarzschild-type coordinate system, rather than an isotropic one, for the transverse space. Defining

$$\begin{aligned}
 R^3 &= \frac{1}{2}(r^3 - 2r_H^3 \pm r^{\frac{3}{2}}\sqrt{r^3 - 4r_H^3}), \\
 f(r) &\equiv 1 - \frac{4r_H^3}{r^3}.
 \end{aligned}
 \tag{2.14}$$

we find that the κ -shell solutions take the more suggestive form

$$\begin{aligned}
 dS_{10}^2 &= -f(r)e^{-\frac{1}{2}(X_0+X_4)}dt^2 + e^{\frac{1}{2}(X_0+X_4)}\left(\frac{dr^2}{f(r)} + r^2d\Omega_4^2\right) + e^{\frac{1}{2}(X_0-X_4)}ds_{K3}^2, \\
 e^{X_4} &\equiv \alpha - (\alpha - 1)f(r), \\
 e^{X_0} &\equiv f^{-\frac{1}{2}\kappa}(\beta - (\beta - 1)f^{\kappa+1}).
 \end{aligned}
 \tag{2.15}$$

In order to be confident that these supergravity solutions are valid, we need to know that the ten-dimensional string-frame geometry has small curvature (in string units) and small dilaton. For the geometries which have enhançons, supergravity is valid all the way in to the shell. These conclusions hold unless we were to try to take a decoupling limit: in that case, the supergravity solution breaks down over a significant domain of the geometry. This is the reason why there is no clean duality between $\mathcal{N} = 2$ gauge theory with no hypermultiplets and this enhançon geometry.

2.2 Relationship to fractional branes

In a related context, the geometry of fractional Dp -branes was studied [9]. Fractional branes can be described as regular $D(p + 2)$ -branes wrapped on a vanishing two-cycle inside the T^4/\mathbb{Z}_2 orbifold limit of K3. The dual gauge theory is again $\mathcal{N} = 2$ SYM with no hypermultiplets. Attempting to take the decoupling limit once again fails to yield a clean strong/weak duality. This happens in a way directly analogous to the original enhançon case.

The authors of [9] found supergravity solutions for fractional branes in six dimensions using two different methods. First, they used boundary state technology to produce a consistent truncation of Type II supergravity coupled to fractional brane sources; second, they related their consistent truncation to the heterotic theory via a chain of dualities. The BPS solutions they found exhibit repulson-like behaviour and an analogous enhançon phenomenon occurs.

The natural extension of this work was, again, to consider the systems when energy is added to take them above the BPS bound. In [6], a consistent six-dimensional truncation ansatz for fractional Dp -branes in orbifold backgrounds was provided, for general $p = 0, 1, 2, 3$. Solutions corresponding to the geometry of non-BPS fractional branes were found, in analogy to the non-BPS enhançon work [4]. After imposition of positivity of ADM mass, half of the solutions were disposed of. One of the remaining solutions was discarded because it did not have a BPS limit.

Considering the other branch (which we will call the shell branch, by obvious analogy), those authors concluded that these geometries will always have an enhançon shell

at arbitrary mass above extremality. Thus they concluded that horizons never form, and that the gauge dual of this phenomenon is also prevented from occurring. In other words, the mass density of these solutions was thought to be bounded such that it is never high enough to form a black hole.

The construction of fractional brane geometries that exhibit the enhançon mechanism is expected to be dual (through T-duality of type IIA on K3) to the original enhançon geometries [2, 9, 6]. However, in view of work reviewed in the previous subsection, the conclusion that horizons never form in the non-BPS fractional brane geometries is puzzling.

To further probe the apparent discord in the behaviour of these two dual systems, let us consider the energy density of the shell solutions for the fractional brane geometries. To do this, we match the exterior metric of the shell branch with a black hole interior to the shell, as before. For definiteness, we pick the fractional D2-brane:

$$\begin{aligned}
 ds_+^2 &= -H(r)^{-\frac{3}{4}} f(r) dt^2 + H(r)^{\frac{1}{4}} \left(\frac{1}{f(r)} dr^2 + r^2 d\Omega_4^2 \right), \\
 ds_-^2 &= -H(r_i)^{-\frac{3}{4}} \frac{f(r_i)}{F(r_i)} F(r) dt^2 + H(r_i)^{\frac{1}{4}} \left(\frac{1}{F(r)} dr^2 + r^2 d\Omega_4^2 \right).
 \end{aligned}
 \tag{2.16}$$

Then, at the shell, which is at incision radius r_i , we get an energy density

$$\rho \sim -\frac{H'}{H} + \frac{8}{r_i} \left(\sqrt{\frac{F}{f}} - 1 \right).
 \tag{2.17}$$

Near the BPS limit, the energy density of this shell branch does *not* have a dominant negative contribution. This is to be contrasted with the previous study of shell branch solutions in the Type IIA on K3 theory relevant to the enhançon. In fact, ρ can be positive or negative for the fractional brane case, depending on how the energy above extremality localizes itself.

We will show that this apparent discord is actually an artifact. The hot fractional brane system exhibits the exact dual behavior to that of the hot enhançon. In particular, we will show that the solutions of [6] are related by duality to the hot enhançon solutions of [4]. By continuously varying the K3 moduli away from the orbifold point, we can reach solutions in which the shell branch solutions once again violate the WEC. In the following sections we pin down the precise map between the two setups, and resurrect the horizon branch on the fractional brane side. We will also exhibit the fractional brane equivalent of the κ -shell solutions.

In order to do this we first embed the D4 brane enhançon solutions in the full six dimensional supergravity describing type IIA string theory compactified on K3. We then show how to generate a complete T-duality orbit of solutions (ie. with arbitrary charges for the six dimensional black hole compatible with an enhançon mechanism, representing any suitable choice of wrapped branes.) We also allow arbitrary values of the K3 moduli at infinity - in the case of wrapped D4 enhançons this is a slight generalisation in that we can also include flat B -fields along the internal directions of the K3.

In order to embed the non-extremal D4 brane solutions of [5] in the six dimensional supergravity, we display a simple two charge truncation which describes the solutions studied in [5]. These solutions can then be lifted straight across into the larger supergravity

theory. In deriving the truncation, it is convenient to switch to heterotic variables using the well-known duality between type IIA on $K3$ and heterotic strings on T^4 . This is also convenient for comparing with the fractional brane solutions of [6] since that paper presents solutions in the heterotic frame. However, we should stress that we are performing T-dualities between different IIA solutions and in principle we could have worked in IIA variables throughout.

3. Six-dimensional supergravity

3.1 Formalism

The massless fields of heterotic string theory compactified on a four-torus (or type IIA on $K3$) are the metric $g_{\mu\nu}$, the B-field $B_{\mu\nu}$, 24 $U(1)$ gauge fields $A_\mu^{(a)}$, ($a = 1 \dots 24$), the dilaton ϕ and a matrix of scalar fields M satisfying:

$$M^T = M, \quad M^T L M = L. \tag{3.1}$$

L is a symmetric matrix which defines an inner product on $\mathbb{R}^{4,20}$. The effective action describing the dynamics of the supergravity fields in six dimensions² is given by

$$S \sim \int d^6x \sqrt{-G} e^{-2\phi} \left[R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - F_{\mu\nu}^{(a)} (L M L)_{ab} F^{(b)\mu\nu} + \frac{1}{8} \text{Tr}(\partial_\mu M L \partial^\mu M L) \right], \tag{3.2}$$

where

$$\begin{aligned} F_{\mu\nu}^{(a)} &= \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} \\ H_{\mu\nu\rho} &= (\partial_\mu B_{\nu\rho} + 2A_\mu^{(a)} L_{ab} F_{\nu\rho}^{(b)}) + \text{cyclic permutations of } \mu, \nu, \rho. \end{aligned} \tag{3.3}$$

The equations of motion for $A_\mu^{(a)}$ lead to the conserved electric charges:

$$v^{(a)} = \int_{S^5} e^{-2\phi} (L M L)_{ab} * F^{(b)}. \tag{3.4}$$

In the classical supergravity theory, these charges can take arbitrary values, but in the quantum theory they are constrained to lie on a lattice $\Gamma^{4,20} \subset \mathbb{R}^{4,20}$. (In the heterotic string the 24 quantized charges are carried by fundamental string states. They are 4 momenta and 4 winding numbers along the T^4 and 16 $U(1)$ charges in the Cartan subalgebra of the 10d gauge group. In IIA strings on $K3$, the charges label integer homology classes in the 24 dimensional $H_*(K3, \mathbb{Z})$. Branes wrapped on cycles carry these charges.)

The effective action (3.2) is invariant under an $O(4,20)$ symmetry group which acts as

$$M \rightarrow \Omega M \Omega^T, \quad A_\mu^{(a)} \rightarrow \Omega_{ab} A_\mu^{(b)}, \quad G_{\mu\nu} \rightarrow G_{\mu\nu}, \quad B_{\mu\nu} \rightarrow B_{\mu\nu}, \quad \phi \rightarrow \phi. \tag{3.5}$$

²in conventions standard for the heterotic theory.

This extends to a symmetry of the full string theory if it also acts on the lattice $\Gamma^{4,20}$. With $\Gamma^{4,20}$ fixed, the discrete subgroup of lattice automorphisms $O(4, 20; \mathbb{Z})$ forms the T-duality group. The action of $\Omega \in O(4, 20)$ on $v^{(a)}$ is

$$v^{(a)} \rightarrow (\Omega^T)_{ab}^{-1} v^{(b)}. \quad (3.6)$$

The scalar matrix M labels the different vacua of the theory³. It will be useful to have a geometrical interpretation of this matrix. First of all, any M of the form obtained via dimensional reduction from $D = 10$ can be written as

$$M = \Omega_0^T \Omega_0, \quad (3.7)$$

for some $\Omega_0 \in O(4, 20)$. The choice of Ω_0 is unique up to left multiplication by an element of $O(4) \times O(20)$. Thus choices of M are labeled by points in

$$\frac{O(4, 20)}{O(4) \times O(20)}. \quad (3.8)$$

This is the space of positive-definite four-planes in $\mathbb{R}^{4,20}$.

Let us see this correspondence more directly. Planes are in one-to-one correspondence with the projection operator onto the plane. Let P_+ be the projection operator onto a positive four-plane. One such projection operator is given by:

$$P = \frac{1}{2}(1_{24} + L) \quad (3.9)$$

and all others are related by:

$$P_+ = \Omega_0^{-1} P \Omega_0, \quad (3.10)$$

for some $\Omega_0 \in O(4, 20)$. So we find that P_+ is related to M as:

$$P_+ = \frac{1}{2}(1_{24} + LM). \quad (3.11)$$

This geometrical language is particularly convenient for expressing the mass of BPS charged states. The charge of a state is labeled by a vector $v^{(a)}$ in the lattice $\Gamma^{4,20}$, as above. The BPS mass depends on the scalars M , and is simply the length of the projection of v onto the four-plane defined by M :

$$m^2 \sim v \cdot P_+ v = v^T L P_+ v. \quad (3.12)$$

Note that this mass formula is invariant under $O(4, 20)$ transformations.

We shall be particularly interested in BPS states which are massive at generic points in moduli space (generic M) but become massless at special enhancement loci. These states correspond to vectors in the charge lattice $\Gamma^{4,20}$ of negative length:

$$v^T L v < 0. \quad (3.13)$$

They become massless when they are orthogonal to the four-plane defined by M

$$P_+ v = 0. \quad (3.14)$$

³For IIA on $K3$ it describes the Kähler and complex structure moduli of the $K3$ as well as flat B -field components in the internal space. For heterotic compactifications we shall be more explicit about the relation of M to 10-dimensional quantities in the following.

3.2 Generating solutions

We would like to generate the widest class of static, spherically symmetric (non-BPS) enhançon-like solutions of the six dimensional supergravity with action (3.2). In order to do this, we should find solutions with arbitrary asymptotic values for the scalar moduli M , and with arbitrary charge vector $v^{(a)}$, subject only to the condition (3.13) which is necessary so that the state can become massless at special points in moduli space. We are looking for solutions representing point-like sources, rather than string-like ones in six dimensions. The restriction of spherical symmetry therefore rules out the six-dimensional $B_{(2)}$ being turned on. The charge vectors we take to be arbitrary. Where we have to make an ansatz in the form of our consistent truncation of supergravity is in taking only two scalar fields to be excited. As with other systems, we expect that horizon branch (black hole) solutions will be unique.

Our main tool for generating solutions will be the $O(4, 20)$ symmetry (3.5). Indeed, given a solution with arbitrary (constant) asymptotic value for M , we can transform it into a solution with $M = 1$ asymptotically by an $O(4, 20)$ transformation and so we can restrict attention to such solutions.

Furthermore, having fixed $M = 1$ asymptotically we still have the freedom to make transformations in the $O(4) \times O(20)$ subgroup of $O(4, 20)$ which fixes the identity matrix. An $O(4)$ rotation can be used to fix the direction of the component of v in the four-plane defined by $M = 1$, whilst an $O(20)$ rotation can be used to fix the direction of the component of v orthogonal to the four-plane. After fixing these directions, we are left with a two parameter family of possible boundary conditions given by the magnitudes of these two components of v .

It will be helpful at this stage to recall the relation between the six-dimensional supergravity and the ten-dimensional heterotic theory. We perform the dimensional reduction using the conventions of Sen [10]. After compactification, the massless six dimensional fields are as follows. There are scalar fields $\hat{G}_{ij}, \hat{B}_{ij}, \hat{A}_i^I$ ($i, j = 1 \dots 4$) ($I = 1 \dots 16$), coming from the internal components of the metric, B -field and $U(1)^{16}$ gauge fields. These are conveniently assembled into the matrix M :

$$M = \begin{pmatrix} \hat{G}^{-1} & \hat{G}^{-1}\hat{D} - 1_4 & \hat{G}^{-1}\hat{A} \\ \hat{D}^T\hat{G}^{-1} - 1_4 & \hat{D}^T\hat{G}^{-1}\hat{D} & \hat{D}^T\hat{G}^{-1}\hat{A} \\ \hat{A}^T\hat{G}^{-1} & \hat{A}^T\hat{G}^{-1}\hat{D} & \hat{A}^T\hat{G}^{-1}\hat{A} + 1_{16} \end{pmatrix}, \quad (3.15)$$

where \hat{G}, \hat{B} and \hat{A} are the matrices with elements $\hat{G}_{ij}, \hat{B}_{ij}$ and \hat{A}_i^I respectively and we have defined $\hat{D} = (\hat{B} + \hat{G} + \frac{1}{2}\hat{A}\hat{A}^T)$.

There is also a six dimensional dilaton, related to the ten dimensional one by

$$e^{-2\phi} = e^{-2\phi^{(10)}} \sqrt{\det \hat{G}}. \quad (3.16)$$

The six dimensional metric $G_{\mu\nu}$ is defined by the relation:

$$dS_{10}^2 = G_{\mu\nu} dx^\mu dx^\nu + \hat{G}_{ij} (dz^i + 2A_\mu^{(i)} dx^\mu) (dz^j + 2A_\nu^{(j)} dx^\nu), \quad (3.17)$$

which also introduces four $U(1)$ gauge fields $A_\mu^{(i)}$. The remaining 20 $U(1)$ gauge fields are given by:

$$A_\mu^{(I+8)} = -\left(\frac{1}{2}A_\mu^{(10)I} - \hat{A}_i^I A_\mu^{(i)}\right), \quad A_\mu^{(i+4)} = \frac{1}{2}B_{i\mu}^{(10)} - \hat{B}_{ij}A_\mu^{(j)} + \frac{1}{2}\hat{A}_j^I A_\mu^{(I+8)}. \quad (3.18)$$

Finally, the six dimensional B-field is given by:

$$B_{\mu\nu} = B_{\mu\nu}^{(10)} - 4\hat{B}_{ij}A_\mu^{(i)}A_\nu^{(j)} - 2(A_\mu^{(i)}A_\nu^{(i+4)} - A_\nu^{(i)}A_\mu^{(i+4)}). \quad (3.19)$$

The charge of a fundamental string state is given by momenta and winding numbers on T^4 and charges under $U(1)^{16}$. We label these charges by $v = (n_i, w^i, q^I)$. The lattice inner product in terms of these charges is

$$v^T L v = 2n_i w^i - q^I q^I. \quad (3.20)$$

In other words, the inner product is

$$L = \begin{pmatrix} 0 & 1_4 & 0 \\ 1_4 & 0 & 0 \\ 0 & 0 & -1_{16} \end{pmatrix} \quad (3.21)$$

in this basis.

Following the discussion above, we should look for a two-charge truncation of the six-dimensional supergravity. A particular choice of state which has ($v^T L v < 0$) is given by a fundamental string with $n_4 = -w_4 = 1$, i.e. one unit of momentum and minus one unit of winding number along the z^4 direction of the torus. This motivates making a ten dimensional ansatz in which only the fields which couple to such a state are turned on.

The truncated supergravity arises if we turn off all of the $U(1)^{16}$ gauge fields of the ten dimensional theory and further require that three of the compactified dimensions are flat space with no fields turned on⁴. We then make an ordinary S^1 reduction on the final compactified direction. Writing this out explicitly, our reduction ansatz is:

$$\begin{aligned} dS_{10}^2 &= G_{\mu\nu} dx^\mu dx^\nu + (dz_1^2 + dz_2^2 + dz_3^2) + \hat{G}_{44}(dz^4 + 2A_\mu^{(4)} dx^\mu)^2, \\ A_\mu^{(8)} &= \frac{1}{2}B_{4\mu}, \quad B_{\mu\nu} = B_{\mu\nu}^{(10)} - 2(A_\mu^{(4)} A_\nu^{(8)} - A_\nu^{(4)} A_\mu^{(8)}), \\ e^{-2\phi} &= e^{-2\phi^{(10)}} \sqrt{\hat{G}_{44}}. \end{aligned} \quad (3.22)$$

The six dimensional field content is thus $(G_{\mu\nu}, B_{\mu\nu}, \phi)$, two gauge fields $A_\mu^{(4)}, A_\mu^{(8)}$ and a scalar field $\hat{G}_{44} \equiv e^K$. Substituting into (3.2) produces the following action for the truncated theory:

$$S \sim \int d^6x \sqrt{-G} e^{-2\phi} \left[R + 4\partial_\mu \phi \partial^\nu \phi - \frac{1}{12} H^2 \right]$$

⁴This corresponds to smearing the string along the three remaining torus directions

$$-\left. \left(e^K (F^{(4)})^2 + e^{-K} (F^{(8)})^2 - \frac{1}{4} (\partial_\mu K \partial^\mu K) \right) \right]. \quad (3.23)$$

By construction, any solution of this theory is also a solution of the full six dimensional supergravity described by the action (3.2). Practically, we will truncate further by setting $B_{\mu\nu} = 0$. This is because we are looking for particle-like solutions in six dimensions (rather than, for example, string-like ones).

Next, we describe how to generate $O(4, 20)$ families of solutions from a given solution of the truncated theory (3.23). First it is useful to introduce a basis for $\mathbb{R}^{4,20}$ in which the inner product L is diagonal. Defining Q to be the orthogonal matrix

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} \cdot 1_4 & \frac{1}{\sqrt{2}} \cdot 1_4 & 0 \\ -\frac{1}{\sqrt{2}} \cdot 1_4 & \frac{1}{\sqrt{2}} \cdot 1_4 & 0 \\ 0 & 0 & 1_{16} \end{pmatrix} \quad (3.24)$$

it is easy to see that the transformation

$$L \rightarrow QLQ^T \quad (3.25)$$

puts L into the diagonal form $L = \text{diag}(1_4, -1_{20})$. We should also transform the matrix M via $M \rightarrow QMQ^T$ and the $U(1)$ gauge fields via $A^{(a)} \rightarrow Q_{ab}A^{(b)}$. The action (3.2) is invariant under this set of transformations

We now introduce a useful notation for embedding solutions of the truncated theory (3.23) into the theory (3.2) written in the new basis. Define a 24-entry column vector by

$$(v^0)^T = (v_L^0, v_R^0) \quad (3.26)$$

where

$$(v_L^0)^T = (0, 0, 0, 1) \quad \text{and} \quad (v_R^0)^T = (0, 0, 0, 1, 0, 0) \quad (3.27)$$

are 4- and 20-vectors respectively. A solution of the truncated theory (3.23) gives rise to a solution of the full six dimensional theory (3.2) with

$$M = 1_{24} + \begin{pmatrix} (\cosh K - 1)v_L^0 v_L^{0T} & (\sinh K)v_L^0 v_R^{0T} \\ (\sinh K)v_R^0 v_L^{0T} & (\cosh K - 1)v_R^0 v_R^{0T} \end{pmatrix},$$

$$F^{(a)} = \begin{pmatrix} (F^L)v_L^0 \\ (F^R)v_R^0 \end{pmatrix}^{(a)}, \quad (3.28)$$

where

$$F^L = \frac{1}{\sqrt{2}}(F^{(4)} + F^{(8)}) \quad , \quad F^R = \frac{1}{\sqrt{2}}(F^{(8)} - F^{(4)}). \quad (3.29)$$

We are interested in solutions for which $K \rightarrow 0$ asymptotically so that $M \rightarrow 1$. We can also shift ϕ by a constant if necessary so that $\phi = 0$ asymptotically. The $U(1)$ charge of the solution is then computed using (3.4) and we find

$$v = \begin{pmatrix} q_L v_L^0 \\ q_R v_R^0 \end{pmatrix}, \quad (3.30)$$

where we have defined

$$q_L = \int_{S^5\{r=\infty\}} *F^L \tag{3.31}$$

and similarly for q_R .

It is straightforward to apply $O(4) \times O(20)$ transformations to these solutions to generate families of solutions with different v . Such transformations have the form

$$\Omega = \begin{pmatrix} R_4(v_L) & 0 \\ 0 & R_{20}(v_R) \end{pmatrix} \tag{3.32}$$

where $R_4(v_L)$ is a 4×4 rotation matrix which rotates the vector v_L^0 into an arbitrary unit length 4-vector v_L and, likewise, $R_{20}(v_R)$ is a 20×20 rotation which takes the vector v_R^0 into an arbitrary unit 20-vector v_R .⁵

After applying the symmetry transformation $M \rightarrow \Omega M \Omega^T$, $F^{(a)} \rightarrow \Omega_{ab} F^{(b)}$, we generate the solution

$$M = 1_{24} + \begin{pmatrix} (\cosh K - 1)v_L v_L^T & (\sinh K)v_L v_R^T \\ (\sinh K)v_R v_L^T & (\cosh K - 1)v_R v_R^T \end{pmatrix},$$

$$F^{(a)} = \begin{pmatrix} (F^L)v_L \\ (F^R)v_R \end{pmatrix}^{(a)}. \tag{3.33}$$

The charge of this solution is

$$v = \begin{pmatrix} q_L v_L \\ q_R v_R \end{pmatrix}. \tag{3.34}$$

The masslessness condition which determines the enhançon radius is

$$0 = P_+ v = \frac{1}{2} [(\cosh K + 1)q_L + \sinh K q_R] L \left(\begin{pmatrix} v_L \\ \left[\frac{\cosh K - 1}{\sinh K} \right] v_R \end{pmatrix} \right), \tag{3.35}$$

or, more simply,

$$(1 + \cosh K)q_L + \sinh K q_R = 0. \tag{3.36}$$

Finally, we can generate further solutions with arbitrary (constant) asymptotic values for M , by acting on the solutions (3.33) with the remaining symmetry transformations in $O(4, 20)/(O(4) \times O(20))$. These transformations act transitively on the space of constant asymptotic values for M .

4. Revisiting hot fractional brane physics

We now return to the explicit solutions of [5] which were reviewed in section 1. The formalism of the previous section allows us to rewrite them in a T-duality covariant way. We then transform to the variables used for the fractional brane solutions and recover and extend the solutions of [6].

⁵Different choices of $R_4(v_L)$ and $R_{20}(v_R)$ for fixed v_L, v_R act identically on the solution (3.28).

We start from the form of the metric (2.3) discussed in section 1. Reducing to six dimensions and then applying S-duality([11]) brings us to the following solution written in the (heterotic) variables of the last section:

$$\begin{aligned}
 dS_6^2 &= -f(r)e^{-(X_0+X_4)}dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_4^2, \\
 \phi_6 &= -\frac{1}{4}(X_0 + X_4), \\
 F_{rt}^L &= -\frac{1}{\sqrt{2}r^4}(q_4e^{-2X_0} + Q_4e^{-2X_4}), \\
 F_{rt}^R &= -\frac{1}{\sqrt{2}r^4}(-q_4e^{-2X_0} + Q_4e^{-2X_4}), \\
 K &= X_0 - X_4,
 \end{aligned} \tag{4.1}$$

A few clarifying comments on the solution-generating process are in order. We have previously indicated that $q_4 = -V_*Q_4/V$, but from the point of view of supergravity these two charges are not related. That is to say, the supergravity solution we are considering solves the equations of motion for any value of Q_4 and q_4 and we have two independent charges. We only find out about the relation between the charges by, for example, probing with a D-brane, a stringy microscopic object. Thus these solutions provide a suitably general, two-charge family of ‘seed’ solutions for generating the full orbit of solutions. After performing $O(4, 20)$ transformations, we can restore the correct quantization condition on the charges by hand.

Now we want to see how the general non-BPS enhançon solutions look in the language of the fractional brane constructions. This will also allow us to confirm that the old solutions of [4] and of [6] are, in fact, related by duality. The appendix contains tedious details of this calculation. The result of transforming the κ -shell solutions to the fractional brane frame is

$$\begin{aligned}
 dS_6^2 &= -fH^{-\frac{1}{2}}dt^2 + H^{\frac{1}{2}}(f^{-1}dr^2 + r^2d\Omega_4^2), \\
 e^{\phi_6} &= H^{\frac{1}{4}}, \\
 G_{aa} &= \frac{\sqrt{H}}{h_1}, \\
 \sqrt{2}D &= \frac{q_2}{q_1} \left(\frac{h_2}{h_1} - 1 \right), \\
 C_t &= -\frac{q_2}{2r^3} \frac{1}{H} \left(h_1 + h_2 - \frac{q_1}{2q_2a} \left(\frac{q_2}{q_1}a - 1 \right)^2 e^{X_4}(e^{X_0}I(r) - 1) \right), \\
 A_t &= -\frac{q_1}{2r^3} \frac{1}{h_1} \left(\frac{q_2}{q_1}a + 1 - \left(\frac{q_2}{q_1}a - 1 \right) e^{X_0}I(r) \right),
 \end{aligned} \tag{4.2}$$

where we have used the shorthand

$$a = -\frac{q_1}{\sqrt{q_2^2 + 2q_1^2}} \tag{4.3}$$

and defined the functions

$$h_1 \equiv \frac{1}{2} \left(\left(\frac{q_2}{q_1}a + 1 \right) e^{X_0} - \left(\frac{q_2}{q_1}a - 1 \right) e^{X_4} \right),$$

$$\begin{aligned}
 h_2 &\equiv \frac{1}{2} \frac{q_1}{q_2 a} \left(\left(\frac{q_2}{q_1} a + 1 \right) e^{X_0} + \left(\frac{q_2}{q_1} a - 1 \right) e^{X_4} \right), \\
 H &\equiv \frac{1}{2a^2} h_1^2 - \frac{q_2^2}{2q_1^2} h_2^2,
 \end{aligned}
 \tag{4.4}$$

and

$$I(r) = -3r^3 \int dr \frac{e^{-2X_0}}{r^4}.
 \tag{4.5}$$

The only property of the latter function that we will need here is that as $\kappa \rightarrow 0$, $I(r) \rightarrow e^{-X_0}$. The fractional brane frame constants a, q_1, q_2 are related to the parameters familiar from the enhançon frame by

$$\begin{aligned}
 Q_4 &= -\frac{q_1}{6} \left(\frac{q_2}{q_1} + \frac{1}{a} \right), \\
 q_4 &= -\frac{q_1}{6} \left(\frac{q_2}{q_1} - \frac{1}{a} \right).
 \end{aligned}
 \tag{4.6}$$

Taking the $\kappa \rightarrow 0$ limit gives the hot fractional brane solutions of [6]. In other words, the latter are none other than the first class of solutions found in the hot enhançon papers [4]. Explicitly, for $\kappa = 0$,

$$\begin{aligned}
 dS_6^2 &= -\frac{f dt^2}{\sqrt{H}} + \sqrt{H} (f^{-1} dr^2 + r^2 d\Omega_4^2), \\
 e^\phi &= H^{\frac{1}{4}}, \\
 G_{aa} &= \frac{H^{\frac{1}{2}}}{h_1}, \quad a = 6, 7, 8, 9, \\
 D &= \frac{q_2}{\sqrt{2}q_1} \left(\frac{h_2}{h_1} - 1 \right), \\
 C_t &= -\frac{q_2 h_3}{H r^3}, \\
 A_t &= -\frac{q_1}{h_1 r^3},
 \end{aligned}
 \tag{4.7}$$

where

$$\begin{aligned}
 H &= \left(1 + \frac{1}{2} \frac{q_2^2}{q_1^2} \right) h_1^2 - \frac{1}{2} \frac{q_2^2}{q_1^2} h_2^2, \\
 h_1 &= 1 - \left(\frac{r_1}{r} \right)^3, \\
 h_2 &= 1 - \left(\frac{r_2}{r} \right)^3, \\
 h_3 &= \frac{1}{2} (h_1 + h_2), \\
 f(r) &= 1 - \left(\frac{r_0}{r} \right)^3, \\
 r_1^3 &= \frac{1}{2} r_0^3 + \frac{1}{2} \frac{\epsilon_1}{\sqrt{q_2^2 + 2q_1^2}} \left[2q_1^4 + (q_1^2 + q_2^2) r_0^6 - 2\epsilon_2 q_1^2 \Lambda \right]^{\frac{1}{2}},
 \end{aligned}$$

$$\begin{aligned}
 r_2^3 &= \frac{1}{2}r_0^3 + \frac{1}{2}\frac{\epsilon_1}{\sqrt{q_2^2}} [2q_1^4 + (q_1^2 + q_2^2)r_0^6 + 2\epsilon_2 q_1^2 \Lambda]^{\frac{1}{2}}, \\
 \Lambda &= (q_1^2 + (q_1^2 + q_2^2) + \frac{1}{4}r_0^4)^{\frac{1}{2}}.
 \end{aligned}
 \tag{4.8}$$

The constants parameterizing non-extremality in the enhançon frame, defined by

$$H_{\text{nonextremal}} - 1 = \alpha(H_{\text{extremal}} - 1) \tag{4.9}$$

are related to the fractional brane quantities via

$$\begin{aligned}
 \alpha_4^3 &= \frac{1}{q_1} \left(-\frac{r_1^3}{a} + \frac{q_2 r_2^3}{q_1} \right), \\
 \alpha_0^3 &= \frac{1}{q_1} \left(\frac{r_1^3}{a} + \frac{q_2 r_2^3}{q_1} \right).
 \end{aligned}
 \tag{4.10}$$

Our dictionary then tells us that the horizon branch solutions in the fractional brane language are given by the values $\epsilon_1 = \epsilon_2 = -1$, while the shell branch solution corresponds to $\epsilon_1 = \epsilon_2 = +1$.

Recovering the BPS solution is straightforward, by using $r_0 \rightarrow 0$ or equivalently $\alpha_0 \rightarrow 1, \alpha_4 \rightarrow 1$. This gives [9],

$$\begin{aligned}
 dS_6^2 &= -\frac{dt^2}{\sqrt{H_{\text{bps}}}} + \sqrt{H_{\text{bps}}}(dr^2 + r^2 d\Omega_4^2), \\
 e^{\phi_6} &= H_{\text{bps}}^{\frac{1}{4}}, \\
 G_{aa} &= H_{\text{bps}}^{\frac{1}{2}}, \\
 D &= -\frac{q_1}{\sqrt{2}r^3}, \\
 C_t &= H_{\text{bps}}^{-1} - 1, \\
 A_t &= -\frac{q_1}{r^3}.
 \end{aligned}
 \tag{4.11}$$

Since

$$H_{\text{bps}} = 1 + \frac{q_2}{r^3} - \frac{q_1^2}{2r^6}, \tag{4.12}$$

the charges of [9] and [2] are consistently related as

$$\begin{aligned}
 -3q_2 &= q_4 + Q_4, \\
 9q_1^2 &= -2q_4 Q_4.
 \end{aligned}
 \tag{4.13}$$

Correctly, this shows that one of q_4 or Q_4 must be negative – as appropriate for our system for which the second charge is induced from the first.

Again, to be confident that these supergravity solutions are valid, we need to know that the ten-dimensional string-frame geometry has small curvature (in string units) and small dilaton. For the geometries which have enhançons, supergravity is valid all the

way in to the shell. Still there is no clean duality between $\mathcal{N} = 2$ gauge theory with no hypermultiplets and this fractional brane geometry with an enhançon, because taking the decoupling limit ruins the validity of the supergravity geometry exterior to the shell.

It is also satisfying to study a wrapped brane probe in these geometries to see where the enhançon radius occurs. By following the duality map, or directly by looking at a fractional brane probe, one sees that the relevant quantity to study in the fractional brane duality frame is the flux through the vanishing 2-cycle,

$$b = \int_c B_{(2)}^{10d}. \tag{4.14}$$

This vanishes at the enhançon radius. In fact, this expression leads directly to the condition

$$e^{X_0 - X_4}|_{r_e} = \frac{V_\star}{V}, \tag{4.15}$$

which is the familiar condition that we found when probing the geometry in the Type II on K3 frame with a wrapped D4 brane.

5. Discussion

In [5], we and co-authors constructed the most general, static, finite-temperature extensions of the BPS enhançon solutions of six-dimensional supergravity possessing spherical symmetry and only one running modulus: the volume of the K3 on which the D-branes are wrapped. In this paper, we generalized the wrapped D4-brane solutions of [5] to have arbitrary charge vector, i.e. arbitrary combinations of D0, D2, and D4 branes wrapped on various cycles in the K3. We also allowed arbitrary values for the K3 moduli at asymptotic infinity. We next showed that a particular subset (previously discovered by [4]) is equivalent to the hot fractional brane solutions found by [6], and thus we widened this class of solutions. We argued that there is a two-branch structure (horizon *and* shell solutions) in both cases.

The context of this wider class of solutions provides a natural explanation as to why the WEC looks different in the case of hot fractional branes. Namely, that the K3 has been taken to a very special point in moduli space – the orbifold limit – and the mass of a BPS fractional brane probe is fixed. There is no longer the freedom to set that mass to be large at infinity, which led to the dominant contribution to the WEC for the original case of D4-branes wrapped on the K3.

There remain outstanding questions about the stability of these various branches of solutions. Some instability must exist, because the horizon branch solutions, which dominate entropically far above extremality, do not exist below a critical value of the mass. Other solutions, i.e. the shell branch solutions (or other exotics), must take over below that point, and connect properly to the known BPS solutions in the limit. Thus, there must be some unstable mode(s) driving the transition between these different states near the critical mass. It is not clear, however, if such a mode is represented in the bulk supergravity theory.

Another impediment to further progress is the lack of a microphysical model of the D-brane and string sources giving rise to these non-BPS enhanced solutions. Supergravity alone is apparently insufficient to settle a number of questions. In particular, for the regime in which the shell branch solutions exist but the horizon branch ones do not, the issue of how much energy (above extremality) gets distributed on the shell, and how much localizes inside the shell in the form of a black hole and/or a hot gas, is undetermined without knowledge of the microphysics.

There are, however, indications that arbitrary distributions of energy above extremality, between the shell, black hole and hot gas, do not make sense. For example, if the hot enhanced shell system is very near extremality and the above-BPS energy is all put into a black hole in the interior of the shell, the black hole must be tiny. This indicates that its Hawking temperature will be high, which would lead to the wrong equation of state for this nearly-BPS system. This indicates that some kind of phase transition might occur, in which a black hole could not form in the interior until it became larger than some critical size. It would be interesting to know if this physics could be reflected in the physics of the strongly coupled $\mathcal{N} = 2$ gauge theory.

Also, as we have seen, supergravity allows a number of additional parameters, for shell branch solutions with given mass and R-R charges, whose microphysical role is unclear. If we were to allow more scalar fields to be turned on, further unfixed parameters in the solutions might be possible.

One might be tempted to think that these parameters could be fixed by considering the shell-branch solutions when their mass gets near to the critical mass at which the horizon branch first appears. Then an argument for protecting the second law of thermodynamics (as in [12]) might give us some information on them. However, as shown in [7], the jump in entropy between the two branches at this point is discontinuous and so more information beyond supergravity would be required.

Acknowledgments

The authors gratefully acknowledge financial support as follows: D.C.P. from PREA of Ontario; A.W.P. from NSERC and CIAR of Canada, and the Alfred P. Sloan Foundation; G.P. from NSERC and the E.C. Stevens Foundation. We also wish to thank K. Hosomichi and O. Saremi for useful discussions.

A. Re-expressing heterotic solutions

In this appendix we re-express the family of solutions of six dimensional heterotic supergravity, given in equation (4.1), in the variables of the fractional brane solutions. In order to do this, we make use of the explicit duality map between a class of ten-dimensional solutions of the heterotic string on T^4 and IIA solutions on T^4/\mathcal{Z}_2 which is described in the appendix of [9].

The plan is as follows. First, we choose the vectors v_L and v_R in a particular way so that the lift to a ten-dimensional heterotic solution has a suitable form and then we

apply the duality transformation of [9] to find a type IIA solution. Finally we reduce to six dimensions to produce the family of solutions quoted in the main text (4.2).

So we start from the solution (4.1) in terms of which the fields of six dimensional heterotic supergravity are written as:

$$F = \begin{pmatrix} F^L v_L \\ F^R v_R \end{pmatrix}, \quad M = 1 + \begin{pmatrix} (\cosh K - 1)v_L v_L^T & \sinh K v_L v_R^T \\ \sinh K v_R v_L^T & (\cosh K - 1)v_R v_R^T \end{pmatrix}.$$

The trick is picking v_L and v_R correctly. For reasons that will be clear shortly, we choose

$$\begin{aligned} v_L^T &= (0 \ 0 \ 0 \ 1), \\ v_R^T &= (0 \ 0 \ 0 \ \frac{q_2}{q_1} a \ a \ a \ 0 \ \dots \ 0), \end{aligned} \quad (\text{A.1})$$

where we have defined

$$\begin{aligned} (-3q_1)^2 &= -2Q_4 q_4, \\ -3q_2 &= Q_4 + q_4, \\ a &= -\frac{q_1}{\sqrt{q_2^2 + 2q_1^2}}. \end{aligned} \quad (\text{A.2})$$

To get into the correct conventions for the lift to ten dimensions we need to apply the transformation

$$\begin{aligned} F &\rightarrow QF \\ M &\rightarrow QMQ^T \end{aligned} \quad (\text{A.3})$$

where

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \quad (\text{A.4})$$

The result is rather a mess:

$$F \rightarrow \frac{1}{\sqrt{2}} \left(0_3, F^L + \frac{q_2}{q_1} a F^R, 0_3, -F^L + \frac{q_2}{q_1} a F^R, \sqrt{2} a F^R, \sqrt{2} a F^R, 0_{14} \right)^T, \quad (\text{A.5})$$

$$M \rightarrow \begin{pmatrix} 1_3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & M_1 & 0 & M_2 & M_4 & M_4 & \dots & 0 \\ 0 & 0 & 1_3 & 0 & 0 & 0 & \dots & 0 \\ 0 & M_2 & 0 & M_3 & M_5 & M_5 & \dots & 0 \\ 0 & M_4 & 0 & M_5 & 1 + M_6 & M_6 & \dots & 0 \\ 0 & M_4 & 0 & M_5 & M_6 & 1 + M_6 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{A.6})$$

where

$$\begin{aligned}
 M_1 &= \left[\frac{1}{2} \left(\frac{q_2}{q_1} a + 1 \right) e^{\frac{1}{2}(X_0 - X_4)} - \frac{1}{2} \left(\frac{q_2}{q_1} a - 1 \right) e^{\frac{1}{2}(X_4 - X_0)} \right]^2, \\
 M_6 &= \frac{1}{2} a^2 \frac{(e^{X_0} - e^{X_4})^2}{e^{X_0 + X_4}},
 \end{aligned} \tag{A.7}$$

and it can be shown that the other functions M_i are related to these via the following relations:

$$\begin{aligned}
 M_2 &= -M_6, \\
 M_4^2 &= M_1 M_6, \\
 M_1 M_5 &= -M_4(1 + M_6), \\
 M_1 M_3 &= (1 + M_6)^2.
 \end{aligned} \tag{A.8}$$

Now, the expressions for M and F uniquely determine the following scalars and gauge fields:

$$\begin{aligned}
 G_{bb} &= 1, \\
 G_{99} &= M_1^{-1}, \\
 a_9^I &= -\sqrt{\frac{M_6}{M_1}}, \\
 F_{rt}^{(1)9} &= \frac{1}{\sqrt{2}} (F_{rt}^L + \frac{q_2}{q_1} a F_{rt}^R), \\
 F_{rt9}^{(2)} &= \frac{1}{\sqrt{2}} (-F_{rt}^L + \frac{q_2}{q_1} a F_{rt}^R), \\
 F_{rt}^{(3)I} &= a F_{rt}^R,
 \end{aligned} \tag{A.9}$$

for $b=6,7,8$ and $I=1$ and 2 .

In order to complete the lift to ten dimensions, we need to integrate the field strengths to form gauge potentials. This introduces the function $I(r)$ of equation (4.5). Now, we can use the duality transformation [9] to convert to a solution of type IIA on T^4/\mathcal{Z}_2 :

$$\begin{aligned}
 \phi^{(HE)} &= -\phi_6, \\
 g_{\mu\nu}^{(HE)} &= e^{-2\phi_6} g_{\mu\nu}, \\
 G_{66}^{(HE)} &= \sqrt{\frac{G_{77} G_{88}}{G_{66} G_{99}}}, \\
 G_{77}^{(HE)} &= \sqrt{\frac{G_{66} G_{88}}{G_{77} G_{99}}}, \\
 G_{88}^{(HE)} &= \sqrt{\frac{G_{66} G_{77}}{G_{88} G_{99}}}, \\
 G_{99}^{(HE)} &= \sqrt{G_{66} G_{77} G_{88} G_{99}}, \\
 A_\mu^{(HE)9} &= C_\mu, \\
 A_\mu^{(HE)I} + A_\mu^{(HE)I+1} &= \sqrt{2} A_\mu^I,
 \end{aligned}$$

$$A_9^{(HE)I} + A_9^{(HE)I+1} = -\sqrt{2}D^I, \tag{A.10}$$

for $I = 1..16$ in general. Looking carefully, we see that there is a change in conventions that must be applied for consistency with our notation:

$$A_M^I \rightarrow \sqrt{2}A_M^I \tag{A.11}$$

Applying this, we get the following solution:

$$\begin{aligned} dS_6^2 &= -f e^{-\frac{1}{2}(X_0+X_4)} dt^2 + e^{\frac{1}{2}(X_0+X_4)} (f^{-1} dr^2 + r^2 d\Omega_4^2), \\ 4\phi_6 &= X_0 + X_4 \\ G_{aa} &= M_1^{-\frac{1}{2}}, \\ \sqrt{2}D &= a \frac{e^{X_0} - e^{X_4}}{\frac{1}{2} \left(\frac{q_2}{q_1} a + 1 \right) e^{X_0} - \frac{1}{2} \left(\frac{q_2}{q_1} a - 1 \right) e^{X_4}}, \\ C_t &= \frac{-q_1}{4ar^3} \left(\left(\frac{q_2}{q_1} a + 1 \right)^2 e^{-X_4} - \left(\frac{q_2}{q_1} a - 1 \right)^2 I(r) \right), \\ A_t &= \frac{-q_1}{r^3} \left(\frac{\frac{q_2}{q_1} a + 1 - \left(\frac{q_2}{q_1} a - 1 \right) I(r) e^{X_0}}{\left(\frac{q_2}{q_1} a + 1 \right) e^{X_0} - \left(\frac{q_2}{q_1} a - 1 \right) e^{X_4}} \right), \end{aligned} \tag{A.12}$$

where $a = 6, 7, 8, 9$. Note that we have redefined D by a factor of $-\sqrt{2}$, another necessary change of convention c.f. [9]. We can rewrite this solution in a suggestive way:

$$\begin{aligned} dS_6^2 &= -f H^{-\frac{1}{2}} dt^2 + H^{\frac{1}{2}} (f^{-1} dr^2 + r^2 d\Omega_4^2), \\ e^{\phi_6} &= H^{\frac{1}{4}}, \\ G_{aa} &= \frac{\sqrt{H}}{h_1}, \\ \sqrt{2}D &= \frac{q_2}{q_1} \left(\frac{h_2}{h_1} - 1 \right), \\ C_t &= -\frac{q_2}{2r^3} \frac{1}{H} \left(h_1 + h_2 - \frac{q_1}{2q_2 a} \left(\frac{q_2}{q_1} a - 1 \right)^2 e^{X_4} (e^{X_0} I(r) - 1) \right), \\ A_t &= -\frac{q_1}{2r^3} \frac{1}{h_1} \left(\frac{q_2}{q_1} a + 1 - \left(\frac{q_2}{q_1} a - 1 \right) e^{X_0} I(r) \right), \end{aligned} \tag{A.13}$$

where we have defined the functions

$$\begin{aligned} h_1 &\equiv \frac{1}{2} \left(\left(\frac{q_2}{q_1} a + 1 \right) e^{X_0} - \left(\frac{q_2}{q_1} a - 1 \right) e^{X_4} \right), \\ h_2 &\equiv \frac{1}{2} \frac{q_1}{q_2 a} \left(\left(\frac{q_2}{q_1} a + 1 \right) e^{X_0} + \left(\frac{q_2}{q_1} a - 1 \right) e^{X_4} \right), \\ H &\equiv e^{X_0+X_4}, \\ &\equiv \frac{1}{2a^2} h_1^2 - \frac{q_2^2}{2q_1^2} h_2^2. \end{aligned} \tag{A.14}$$

Working out the rest of the physics is taken up in the body of the paper.

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